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FREQUENCY SPECTRA OF SOLUTIONS OF STOCHASTIC EQUATIONS, WITH APPLICATIONS TO SPECTRAL BROADENING OF WAVES PROPAGATING IN A RANDOM MEDIUM

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ABSTRACT

Formulas for the cross-correlation and spectral density functions of the solution of a general linear time-dependent stochastic equation are derived. The analysis is based on a modification of the smoothing method. The formulas are applied to the case of radiation of scalar waves by a random point source in a three-dimensional time-dependent random medium. The medium is assumed to be statistically homogeneous and isotropic and to be statistically independent of the source. An approximate expression for the power spectrum of the wave as a function of the source-field point distance (or propagation distance) is obtained for the case in which the characteristic frequency of the source is much greater than that of the medium. This expression shows that as the propagation distance goes to zero the wave spectrum approaches the source spectrum; whereas as the propagation distance becomes infinite the wave spectrum tends to a limiting form which is referred to here as the fully-developed spectrum. It is also found that the total signal power is conserved as the spectrum evolves. Numerical results obtained for the case of a narrow-band source show a progressive broadening of the wave spectrum with increasing propagation distance and/or with increasing strength of the randomness of the medium, in agreement with observations.

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INTRODUCTION

Broadening of the frequency spectrum of an initially narrow-band wave field is a phenomenon which is characteristic of wave propagation in real, time-dependent media such as the atmosphere or ocean. It is usually a consequence of time variations in the medium through which the wave propagates, but may also be due to interaction of the wave with a time-varying boundary. Spectral broadening has been observed in the case of both acoustic and electromagnetic waves, in which contexts it is most often the result of either propagation through turbulence[1,2] or scattering by the sea surface.[3]

The present investigation was undertaken with the objective of studying spectral broadening from a quite general viewpoint; namely, that of the evolution of frequency spectra of solutions of linear time-dependent stochastic equations. The approach is based on the observation, stated in its most general terms, that an essential element in the evolution of the spectrum of the solution u of an equation having the form of Eq. 1 arises when the operator L does not necessarily commute with translation operators in time. A general analysis incorporating this idea is carried out in Sec. I; the problem of spectral broadening of waves propagating in a random medium is then treated in Sec. II as an application.

The general analysis is based on the smoothing method,[4] modified to include the case in which f (referring again to Eq. 1) as well as L is random. A key assumption is that f and L are statistically independent, which, in terms of spectral broadening of waves, is equivalent to assuming that the source of the waves is statistically independent of the mechanism giving rise to the spectral broadening. This assumption would appear to be a reasonable one in most practical situations of interest.

Previous theoretical investigations of spectral broadening of waves propagating in random media have been carried out by Howe,[5] Fante,[6,7] and Woo, et al.[8] Howe derived a kinetic equation and used it to study the effect of the random velocity field on the frequency spectrum of an acoustic wave propagating in a turbulent fluid. Fante used transport theory to study frequency spectra of beamed waves propagating in a turbulent atmosphere. The analysis of Woo, et al. (see also Ref. 9, p. 422) is based on the parabolic equation. Howe treated the case of an isotropic time-dependent turbulence field, whereas both Fante and Woo, et al. assumed that the time variations of the medium were the result of a steady mean wind convecting a "frozen" turbulence field in a direction transverse to the beam. The results of both Howe and Fante indicate that the characteristic width of the wave spectrum increases as some power of the propagation distance. The results of Woo, et al. are given in a somewhat more complicated form, but seem to show a similar effect. In contrast, the present results indicate that, at least in the case of high-frequency waves, the spectrum approaches a limiting form (the fully-developed spectrum) with increasing propagation distance.

The problem of spectral broadening has also been discussed from a theoretical viewpoint by Adomian,[10] and has been treated in a recent paper by Kuznetsova and Chernov.[11] Related work, concerned mainly with spectra of scattered waves and with spectra of amplitude and phase fluctuations of waves propagating in random media, can be found in Refs. 12 - 18.

I. GENERAL FORMULATION

A. Equations for $\bar{u}$ and $\tilde{u}$

We begin by considering a general stochastic equation of the form

$$Lu = f , \tag{1}$$

where L is a linear operator on a vector space, u is an unknown vector, and f is a given vector. The operator L is assumed to be random; i.e., L is assumed to depend on a parameter a which is an element of a sample space A. The space A, together with a σ-algebra of subsets and a probability measure, forms a probability space. The vector f is also random; however, f is assumed to be statistically independent of L. Thus, f may be regarded as being dependent upon a parameter b ranging over a different sample space B which, together with its own σ-algebra of subsets and probability measure, also forms a probability space.

It is clear that the solution u of Eq. 1, as well as functions of it, will depend on both a and b. (The dependence on the parameters a and b of the various quantities appearing in this analysis will not, in general, be explicitly indicated.) It will be necessary, therefore, in what follows to distinguish between ensemble averages over the space A, which will be denoted by $\langle\ \rangle_A$, and averages over B, denoted by $\langle\ \rangle_B$. An average over both A and B (i.e., an ensemble average over the product sample space $A \times B$) will be denoted simply by $\langle\ \rangle$. We note that generally $\langle\ \rangle = \langle\langle\ \rangle_A\rangle_B = \langle\langle\ \rangle_B\rangle_A$.

In order to solve Eq. 1 it is convenient to assume that L depends on a real parameter ε, and that, in a neighborhood of $\varepsilon = 0$, the expansion

$$L = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \cdots \tag{2}$$

is valid. The operator L_0 is assumed to be determinate (i.e., non-random) with a known inverse; whereas L_1, L_2, etc. are generally assumed to be random. Approximate equations, valid when ε is small, can now be obtained for $\bar{u}$ and $\tilde{u}$, where $\bar{u} \equiv \langle u\rangle_A$ and $\tilde{u} \equiv u - \langle u\rangle_A$. The procedure is entirely analogous to that described by Keller[19] (see also Ref. 4). It is only necessary to keep in mind that since f is independent of a it is unaffected by averaging over A. For the case when $\langle L_1\rangle_A = 0$, which is usually the case in practice, the results are

$$[L_0 + \varepsilon^2(\langle L_2\rangle_A - \langle L_1 L_0^{-1} L_1\rangle_A)]\,\bar{u} = f\,, \tag{3}$$

$$\tilde{u} = -\varepsilon L_0^{-1} L_1 \bar{u}\,. \tag{4}$$

Terms of order ε^3 have been dropped from Eq. 3; terms of order ε^2 have been dropped from Eq. 4.

In the special case in which f is determinate, the quantities $\bar{u}$ (which is then also determinate) and $\tilde{u}$ correspond respectively to the mean and fluctuating fields. In that context the type of approach leading to Eqs. 3 and 4, which involves obtaining separate equations for the mean and fluctuating fields, is referred to as the smoothing method by Frisch.[4]

B. Equations for the correlation and spectral density functions

We consider now the case in which u and f are real-valued scalar functions of $\underset{\sim}{x}$ and t (substantially the same analysis as that which follows applies in the case of vector functions), where $\underset{\sim}{x}\,[= (x_1, \cdots, x_n)]$ is an n-dimensional spatial coordinate and t is time. Only the case of free-space propagation will be considered here, and hence the coordinates

$x_1, \cdots, x_n, t$ are unbounded. We assume that $\langle L_1 \rangle_A = 0$, so that the equations for $\bar{u}$ and $\tilde{u}$ are given by Eqs. 3 and 4. We assume also that $\langle f \rangle_B = 0$.

We shall be concerned in the remainder of this paper only with random processes which are stationary in time. (By stationary we mean stationary in the wide sense; i.e., that correlation functions of the form given by Eq. 16 are independent of t.) In order to ensure that $u(t,\underset{\sim}{x})$ is stationary in time, it will be necessary, in addition to assuming that $f(t,\underset{\sim}{x})$ is stationary in time, to impose some conditions on the operators L_0, L_1, L_2, etc. We shall assume here that the operators $L_0, \langle L_1 L_0^{-1} L_1 \rangle_A$, and $\langle L_2 \rangle_A$ commute with translation operators in time. Then the operator M will also commute with translation operators in time, where M is the operator appearing in Eq. 3; i.e.,

$$M \equiv L_0 + \varepsilon^2(\langle L_2 \rangle_A - \langle L_1 L_0^{-1} L_1 \rangle_A) . \tag{5}$$

That these assumptions are sufficient for our purposes will become clear as the analysis proceeds. It should be noted that these assumptions are not so restrictive as to exclude, for example, the case in which L is a linear differential operator with stationary coefficients. (The question of temporal stationarity of random linear systems has been discussed in more detail by Vasholz.[20])

We assume next that the operator L_0^{-1} can be expressed in terms of a Green's function; i.e., we assume that there exists a function $G_0(t, \underset{\sim}{x}, \underset{\sim}{x}')$ which satisfies, along with appropriate initial and/or boundary conditions, the equation

$$L_0 G_0(t, \underset{\sim}{x}, \underset{\sim}{x}') = \delta(t)\delta(\underset{\sim}{x} - \underset{\sim}{x}') . \tag{6}$$

Then L_0^{-1} can be expressed in the form

$$L_0^{-1} w(t, \underset{\sim}{x}) = \iint G_0(t - t', \underset{\sim}{x}, \underset{\sim}{x}') w(t', \underset{\sim}{x}') dt' d\underset{\sim}{x}', \tag{7}$$

where $w(t, \underset{\sim}{x})$ is any function for which the integral exists. (Here, and henceforth, an integral sign without limits denotes an integral from $-\infty$ to $+\infty$.) Similarly, we assume that the inverse operator M^{-1} exists and that it can be expressed in terms of the Green's function $G(t, \underset{\sim}{x}, \underset{\sim}{x}')$; i.e., we assume that G is a solution of

$$MG(t, \underset{\sim}{x}, \underset{\sim}{x}') = \delta(t)\delta(\underset{\sim}{x} - \underset{\sim}{x}'). \tag{8}$$

Then the solution of Eq. 3 can be written

$$\bar{u}(t, \underset{\sim}{x}) = \iint G(t - t', \underset{\sim}{x}, \underset{\sim}{x}') f(t', \underset{\sim}{x}') \, dt' d\underset{\sim}{x}'. \tag{9}$$

By making a change of integration variable we can write Eqs. 7 and 9 in the alternate form

$$L_0^{-1} w(t, \underset{\sim}{x}) = \iint G_0(t', \underset{\sim}{x}, \underset{\sim}{x}') w(t-t', \underset{\sim}{x}') \, dt' d\underset{\sim}{x}', \tag{10}$$

$$\bar{u}(t, \underset{\sim}{x}) = \iint G(t', \underset{\sim}{x}, \underset{\sim}{x}') f(t-t', \underset{\sim}{x}') \, dt' d\underset{\sim}{x}'. \tag{11}$$

Note that it is the assumption that the operators L_0 and M commute with time translations that allows the Green's functions G_0 and G in Eqs. 7 and 9 to be written as functions of the difference $t - t'$ instead of in the more general form $G_0(t, t', \underset{\sim}{x}, \underset{\sim}{x}')$ and $G(t, t', \underset{\sim}{x}, \underset{\sim}{x}')$. This property of the Green's functions is necessary for the stationarity of u. Note also that both G_0 and G are determinate functions.

Operating with L_1 on $\bar{u}$, as given by Eq. 9, yields

$$L_1\bar{u}(t,\underset{\sim}{x}) = \iint F(t,\underset{\sim}{x},t-t',\underset{\sim}{x}')f(t',\underset{\sim}{x}')dt'd\underset{\sim}{x}', \tag{12}$$

where F is defined by

$$F(t,\underset{\sim}{x},t-t',\underset{\sim}{x}') = L_1G(t-t',\underset{\sim}{x},\underset{\sim}{x}'). \tag{13}$$

The random function F, which depends on a but not on b, has the property that $\langle F\rangle_A = 0$. (This follows from the assumption that $\langle L_1\rangle_A = 0$.) Note that F, as defined by Eq. 13, must generally be written as a function of both t and $t-t'$, and not simply as a function of $t-t'$. In order to justify writing F as a function only of $t-t'$ it would be necessary to assume that the operator L_1 commutes with time translations. This assumption, however, is too restrictive: As the following analysis shows, it is the dependence of F on t, as well as on $t-t'$, which is a key element in the evolution of the frequency spectrum of u. We shall, however, assume that the function $F(t,\underset{\sim}{x},s,\underset{\sim}{y})$ is stationary in t.

By making a change of integration variable we can write Eq. 12 in the form

$$L_1\bar{u}(t,\underset{\sim}{x}) = \iint F(t,\underset{\sim}{x},t',\underset{\sim}{x}')f(t-t',\underset{\sim}{x}')dt'd\underset{\sim}{x}'. \tag{14}$$

Operating on Eq. 14 with L_0^{-1}, as given by Eq. 10, and substituting the result into Eq. 4 yields

$$\begin{aligned}\tilde{u}(t,\underset{\sim}{x}) = -\varepsilon\int\cdots\int &G_0(t',\underset{\sim}{x},\underset{\sim}{x}')F(t-t',\underset{\sim}{x}',t'',\underset{\sim}{x}'')\\ &\times f(t-t'-t'', \underset{\sim}{x}'')dt'd\underset{\sim}{x}'dt''d\underset{\sim}{x}''.\end{aligned} \tag{15}$$

The cross-correlation function $R(\tau,\underset{\sim}{x},\underset{\sim}{y})$ is defined by

$$R(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \langle u(t,\underset{\sim}{x})u(t-\tau,\underset{\sim}{y})\rangle . \tag{16}$$

(In the case in which u is an m-component vector function, the cross-correlation function $R_{ij}(\tau,\underset{\sim}{x},\underset{\sim}{y})$ is defined by

$$R_{ij}(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \langle u_i(t,\underset{\sim}{x})u_j(t-\tau,\underset{\sim}{y})\rangle ;$$

$i=1,\cdots,m$, $j=1,\cdots,m$, where the subscripts on u denote vector components.) Upon writing u as the sum $u=\bar{u}+\tilde{u}$ in Eq. 16 we obtain

$$\begin{aligned} R(\tau,\underset{\sim}{x},\underset{\sim}{y}) &= \langle \bar{u}(t,\underset{\sim}{x})\bar{u}(t-\tau,\underset{\sim}{y})\rangle \\ &+ \langle \bar{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle + \langle \tilde{u}(t,\underset{\sim}{x})\bar{u}(t-\tau,\underset{\sim}{y})\rangle \\ &+ \langle \tilde{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle . \end{aligned} \tag{17}$$

The two cross terms on the right-hand side of Eq. 17; i.e., the terms involving products of $\bar{u}$ and $\tilde{u}$, vanish. This follows from the fact that $\bar{u}$ is independent of a and that $\langle\tilde{u}\rangle_A = 0$. Thus, for the first cross term, we can write

$$\begin{aligned} \langle \bar{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle &= \langle\langle \bar{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle_A\rangle_B \\ &= \langle \bar{u}(t,\underset{\sim}{x}) \langle \tilde{u}(t-\tau,\underset{\sim}{y})\rangle_A\rangle_B = 0 , \end{aligned}$$

and similarly for the second cross term. Expressions for the remaining two terms on the right-hand side of Eq. 17 can be obtained with the aid of Eqs. 11 and 15. These expressions are

$$<\bar{u}(t,\underset{\sim}{x})\bar{u}(t-\tau,\underset{\sim}{y})> = \int\cdots\int G(t',\underset{\sim}{x},\underset{\sim}{x}')G(s',\underset{\sim}{y},\underset{\sim}{y}')$$

$$\times <f(t-t',\underset{\sim}{x}')f(t-\tau-s',\underset{\sim}{y}')>_B dt'd\underset{\sim}{x}'ds'd\underset{\sim}{y}' , \tag{18}$$

$$<\tilde{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})> = \varepsilon^2\int\cdots\int G_0(t',\underset{\sim}{x},\underset{\sim}{x}')G_0(s',\underset{\sim}{y},\underset{\sim}{y}')$$

$$\times <F(t-t',\underset{\sim}{x}',t'',\underset{\sim}{x}'')\,F(t-\tau-s',\underset{\sim}{y}',s'',\underset{\sim}{y}'')>_A <f(t-t'-t'',\underset{\sim}{x}'')$$

$$\times f(t-\tau-s'-s'',\underset{\sim}{y}'')>_B dt'd\underset{\sim}{x}'ds'd\underset{\sim}{y}'dt''d\underset{\sim}{x}''ds''d\underset{\sim}{y}'' . \tag{19}$$

In deriving Eq. 19 use has been made of the fact that F is independent of b and that f is independent of a. The terms on the right-hand sides of Eqs. 18 and 19 can be expressed in terms of the correlation functions Φ and R_0, which are defined by

$$\Phi(\tau,\underset{\sim}{x},\underset{\sim}{y},t',\underset{\sim}{x}',s',\underset{\sim}{y}') = <F(t,\underset{\sim}{x},t',\underset{\sim}{x}')F(t-\tau,\underset{\sim}{y},s',\underset{\sim}{y}')>_A , \tag{20}$$

$$R_0(\tau,\underset{\sim}{x},\underset{\sim}{y}) = <f(t,\underset{\sim}{x})f(t-\tau,\underset{\sim}{y})>_B . \tag{21}$$

Equation 17 can then be written

$$R(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \bar{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) + \tilde{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) , \tag{22}$$

where

$$\bar{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) = <\bar{u}(t,\underset{\sim}{x})\bar{u}(t-\tau,\underset{\sim}{y})> = \int\cdots\int G(t',\underset{\sim}{x},\underset{\sim}{x}')$$

$$\times G(s',\underset{\sim}{y},\underset{\sim}{y}')R_0(\tau-t'+s',\underset{\sim}{x}',\underset{\sim}{y}')dt'd\underset{\sim}{x}'ds'd\underset{\sim}{y}' , \tag{23}$$

$$\tilde{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \langle\tilde{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle = \varepsilon^2\int\cdots\int G_0(t',\underset{\sim}{x},\underset{\sim}{x}')$$

$$\times G_0(s',\underset{\sim}{y},\underset{\sim}{y}')\Phi(\tau-t'+s',\underset{\sim}{x}',\underset{\sim}{y}',t'',\underset{\sim}{x}'',s'',\underset{\sim}{y}'')$$

$$\times R_0(\tau-t'+s'-t''+s'',\underset{\sim}{x}'',\underset{\sim}{y}'')dt'd\underset{\sim}{x}'ds'd\underset{\sim}{y}'dt''d\underset{\sim}{x}''ds''d\underset{\sim}{y}'' . \tag{24}$$

Note that $u(t,\underset{\sim}{x})$, as calculated here, is indeed stationary in time, as can be seen by referring to Eqs. 16 and 22-24.

The spectral density function $S(\omega,\underset{\sim}{x},\underset{\sim}{y})$ is defined by

$$S(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \int R(\tau,\underset{\sim}{x},\underset{\sim}{y})e^{i\omega\tau}d\tau . \tag{25}$$

With the aid of Eq. 22 we can write

$$S(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \bar{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) + \tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) , \tag{26}$$

where $\bar{S}$ and $\tilde{S}$ are defined by

$$\bar{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \int \bar{R}(\tau,\underset{\sim}{x},\underset{\sim}{y})e^{i\omega\tau}d\tau , \tag{27}$$

$$\tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \int \tilde{R}(\tau,\underset{\sim}{x},\underset{\sim}{y})e^{i\omega\tau}d\tau . \tag{28}$$

To calculate $\bar{S}$ we insert into Eq. 27 the expression for $\bar{R}$ given by Eq. 23 and carry out the integration over τ, t', and s'. The result is

$$\bar{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \iint H(\omega,\underset{\sim}{x},\underset{\sim}{x}')H^*(\omega,\underset{\sim}{y},\underset{\sim}{y}')S_0(\omega,\underset{\sim}{x}',\underset{\sim}{y}')d\underset{\sim}{x}'d\underset{\sim}{y}' , \tag{29}$$

where we have defined

$$H(\omega,\underset{\sim}{x},\underset{\sim}{x}') = \int G(t,\underset{\sim}{x},\underset{\sim}{x}')e^{i\omega t}dt\ , \tag{30}$$

$$S_0(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \int R_0(\tau,\underset{\sim}{x},\underset{\sim}{y})e^{i\omega\tau}d\tau\ , \tag{31}$$

and the symbol $(\ )^*$ denotes a complex conjugate. Similarly, an expression for $\tilde{S}$ is obtained by substituting the formula for $\tilde{R}$ given by Eq. 24 into Eq. 28 and carrying out the integration over τ, t', s', t'', and s''. This procedure yields

$$\begin{aligned}\tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) &= (\varepsilon^2/2\pi)\int\cdots\int H_0(\omega,\underset{\sim}{x},\underset{\sim}{x}')H_0^*(\omega,\underset{\sim}{y},\underset{\sim}{y}')\\ &\times\Theta(\omega-\omega',\underset{\sim}{x}',\underset{\sim}{y}',\omega',\underset{\sim}{x}'',-\omega',\underset{\sim}{y}'')S_0(\omega',\underset{\sim}{x}'',\underset{\sim}{y}'')d\omega'\\ &\times d\underset{\sim}{x}'d\underset{\sim}{y}'d\underset{\sim}{x}''d\underset{\sim}{y}''\ ,\end{aligned} \tag{32}$$

where we have defined H_0 and Θ by

$$H_0(\omega,\underset{\sim}{x},\underset{\sim}{x}') = \int G_0(t,\underset{\sim}{x},\underset{\sim}{x}')e^{i\omega t}dt\ , \tag{33}$$

$$\begin{aligned}\Theta(\omega,\underset{\sim}{x},\underset{\sim}{y},\omega_+,\underset{\sim}{x}',\omega_-,\underset{\sim}{y}') &= \int\cdots\int\Phi(\tau,\underset{\sim}{x},\underset{\sim}{y},t',\underset{\sim}{x}',s',\underset{\sim}{y}')\\ &\times e^{i(\omega\tau+\omega_+t'+\omega_-s')}d\tau dt'ds'.\end{aligned} \tag{34}$$

In deriving Eq. 32 we have made use of some known results relating the Fourier transform of a product of two functions to the convolution of the transformed functions.

We consider now the special case in which L_1 can be written as a finite linear combination of the form

$$L_1 = \mu_p(t,\underset{\sim}{x})K_p\ ;\quad p = 1,\cdots,N\ , \tag{35}$$

(summation on repeated indices is implied), where the μ_p's are random functions, stationary in time and with zero mean (i.e., $\langle\mu_p\rangle_A = 0$; $p = 1,\cdots,N$), and the K_p's are determinate operators which commute with time translations. (Of particular interest, of course, is the case in which the K_p's are differential operators with constant coefficients.) Then, using Eq. 13, we can write

$$F(t,\underset{\sim}{x},s,\underset{\sim}{y}) = \mu_p(t,\underset{\sim}{x})G_p(s,\underset{\sim}{x},\underset{\sim}{y}) , \tag{36}$$

where we have defined

$$G_p(t,\underset{\sim}{x},\underset{\sim}{x}') = K_p G(t,\underset{\sim}{x},\underset{\sim}{x}') ; \quad p = 1,\cdots,N . \tag{37}$$

Inserting Eq. 36 into Eq. 20 yields

$$\begin{aligned}\Phi(\tau,\underset{\sim}{x},\underset{\sim}{y},t',\underset{\sim}{x}',s',\underset{\sim}{y}') &= G_p(t',\underset{\sim}{x},\underset{\sim}{x}')G_q(s',\underset{\sim}{y},\underset{\sim}{y}') \\ &\quad \times \Gamma_{pq}(\tau,\underset{\sim}{x},\underset{\sim}{y}) ,\end{aligned} \tag{38}$$

where Γ_{pq} is defined by

$$\Gamma_{pq}(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \langle\mu_p(t,\underset{\sim}{x})\mu_q(t-\tau,\underset{\sim}{y})\rangle_A \tag{39}$$

for $p = 1,\cdots,N, q = 1,\cdots,N$. Upon combining Eq. 34 with Eq. 38 we find that

$$\begin{aligned}\Theta(\omega,\underset{\sim}{x},\underset{\sim}{y},\omega_+,\underset{\sim}{x}',\omega_-,\underset{\sim}{y}') &= H_p(\omega_+,\underset{\sim}{x},\underset{\sim}{x}')H_q(\omega_-,\underset{\sim}{y},\underset{\sim}{y}') \\ &\quad \times Z_{pq}(\omega,\underset{\sim}{x},\underset{\sim}{y}) ,\end{aligned} \tag{40}$$

where we have defined

$$H_p(\omega, \underset{\sim}{x}, \underset{\sim}{x}') = \int G_p(t, \underset{\sim}{x}, \underset{\sim}{x}')e^{i\omega t}dt \tag{41}$$

for $p = 1, \cdots, N$, and

$$Z_{pq}(\omega, \underset{\sim}{x}, \underset{\sim}{y}) = \int \Gamma_{pq}(\tau, \underset{\sim}{x}, \underset{\sim}{y})e^{i\omega\tau}d\tau \tag{42}$$

for $p = 1, \cdots, N$, $q = 1, \cdots, N$. For this case the expression for $\bar{R}$ is the same as that given by Eq. 23; however, the expression for $\tilde{R}$, which is obtained by inserting Eq. 38 into Eq. 24, becomes

$$\begin{aligned} \tilde{R}(\tau, \underset{\sim}{x}, \underset{\sim}{y}) = \varepsilon^2 \int \cdots \int & G_0(t', \underset{\sim}{x}, \underset{\sim}{x}')G_0(s', \underset{\sim}{y}, \underset{\sim}{y}') \\ \times & G_p(t'', \underset{\sim}{x}', \underset{\sim}{x}'')G_q(s'', \underset{\sim}{y}', \underset{\sim}{y}'')\Gamma_{pq}(\tau - t' + s', \underset{\sim}{x}', \underset{\sim}{y}') \\ \times & R_0(\tau - t' + s' - t'' + s'', \underset{\sim}{x}'', \underset{\sim}{y}'')dt'd\underset{\sim}{x}'ds'd\underset{\sim}{y}'dt''d\underset{\sim}{x}''ds''d\underset{\sim}{y}'' . \end{aligned} \tag{43}$$

Similarly, the expression for $\bar{S}$ for this case is the same as that given by Eq. 29, whereas the expression for $\tilde{S}$, which is obtained by substituting Eq. 40 into Eq. 32, becomes

$$\begin{aligned} \tilde{S}(\omega, \underset{\sim}{x}, \underset{\sim}{y}) = (\varepsilon^2/2\pi) \int \cdots \int & H_0(\omega, \underset{\sim}{x}, \underset{\sim}{x}')H_0^*(\omega, \underset{\sim}{y}, \underset{\sim}{y}') \\ \times & \Big[Z_{pq}(\omega, \underset{\sim}{x}', \underset{\sim}{y}') * H_p(\omega, \underset{\sim}{x}', \underset{\sim}{x}'')H_q^*(\omega, \underset{\sim}{y}', \underset{\sim}{y}'') \\ & \times S_0(\omega, \underset{\sim}{x}'', \underset{\sim}{y}'') \Big] d\underset{\sim}{x}'d\underset{\sim}{y}'d\underset{\sim}{x}''d\underset{\sim}{y}'' . \end{aligned} \tag{44}$$

The notation $(\ ) * (\ )$ denotes a convolution with respect to ω; i.e.,

$$f * g(\omega) = \int f(\omega - \omega')g(\omega')d\omega' .$$

Whenever the convolution symbol appears inside brackets, as in Eq. 44, it is to be understood that only the terms inside the brackets are involved in the convolution.

Referring again to the general analysis (i.e., the analysis leading to Eqs. 23, 24, 29, and 32), an important special case occurs when the operators L_0 and M commute with space, as well as time, translations. Then, in an obvious change of notation, we can write the Green's functions $G_0(t-t',\mathbf{x},\mathbf{x}')$ and $G(t-t',\mathbf{x},\mathbf{x}')$ in the form $G_0(t-t',\mathbf{x}-\mathbf{x}')$ and $G(t-t',\mathbf{x}-\mathbf{x}')$, where G_0 and G are solutions of

$$L_0 G_0(t,\mathbf{x}) = \delta(t)\delta(\mathbf{x}) \,, \tag{45}$$

and

$$MG(t,\mathbf{x}) = \delta(t)\delta(\mathbf{x}) \,. \tag{46}$$

In addition, the function F is now defined by

$$F(t,\mathbf{x},t-t',\mathbf{x}-\mathbf{x}') = L_1 G(t-t',\mathbf{x}-\mathbf{x}') \,, \tag{47}$$

and is assumed to be stationary in the first two variables.

By making some changes of integration variables in Eqs. 11 and 15, we can write the expressions for $\bar{u}$ and $\tilde{u}$ for this case in the form

$$\bar{u}(t,\mathbf{x}) = \iint G(t',\mathbf{x}')f(t-t',\mathbf{x}-\mathbf{x}')\,dt'd\mathbf{x}' \,, \tag{48}$$

$$\tilde{u}(t,\mathbf{x}) = -\varepsilon\int\cdots\int G_0(t',\mathbf{x}')F(t-t',\mathbf{x}-\mathbf{x}',t'',\mathbf{x}'') \times f(t-t'-t'',\mathbf{x}-\mathbf{x}'-\mathbf{x}'')\,dt'd\mathbf{x}'dt''d\mathbf{x}'' \,. \tag{49}$$

From Eqs. 48 and 49 expressions for $\bar{R}$ and $\tilde{R}$ can be obtained in a manner similar to that used in deriving Eqs. 23 and 24. The result is

$$\bar{R}(\tau,\mathbf{x},\mathbf{y}) = \int\cdots\int G(t',\mathbf{x}')G(s',\mathbf{y}') \times R_0(\tau-t'+s',\mathbf{x}-\mathbf{x}',\mathbf{y}-\mathbf{y}')\,dt'd\mathbf{x}'ds'd\mathbf{y}' \,, \tag{50}$$

$$\tilde{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \varepsilon^2 \int\cdots\int G_0(t',\underset{\sim}{x}')G_0(s',\underset{\sim}{y}')$$

$$\times\Phi(\tau - t' + s', \underset{\sim}{y} - \underset{\sim}{x} + \underset{\sim}{x}' - \underset{\sim}{y}', t'', \underset{\sim}{x}'', s'', \underset{\sim}{y}'')$$

$$\times R_0(\tau - t' + s' - t'' + s'', \underset{\sim}{x} - \underset{\sim}{x}' - \underset{\sim}{x}'', \underset{\sim}{y} - \underset{\sim}{y}' - \underset{\sim}{y}'')dt'd\underset{\sim}{x}'ds'd\underset{\sim}{y}'$$

$$\times dt''d\underset{\sim}{x}''ds''d\underset{\sim}{y}'' , \qquad (51)$$

where, for this case, the function Φ is defined by

$$\Phi(\tau,\underset{\sim}{\xi},t',\underset{\sim}{x}',s',\underset{\sim}{y}') = \langle F(t,\underset{\sim}{x},t',\underset{\sim}{x}')F(t-\tau,\underset{\sim}{x}+\underset{\sim}{\xi},s',\underset{\sim}{y}')\rangle_A . \qquad (52)$$

Expressions for $\bar{S}$ and $\tilde{S}$ for this case are obtained by inserting the formulas for $\bar{R}$ and $\tilde{R}$ given by Eqs. 50 and 51 into Eqs. 27 and 28. The result is

$$\bar{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \iint H(\omega,\underset{\sim}{x}')H^*(\omega,\underset{\sim}{y}')S_0(\omega,\underset{\sim}{x}-\underset{\sim}{x}',\underset{\sim}{y}-\underset{\sim}{y}')\, d\underset{\sim}{x}'d\underset{\sim}{y}' , \qquad (53)$$

$$\tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = (\varepsilon^2/2\pi)\int\cdots\int H_0(\omega,\underset{\sim}{x}')H_0^*(\omega,\underset{\sim}{y}')$$

$$\times\Theta(\omega-\omega',\underset{\sim}{y}-\underset{\sim}{x}+\underset{\sim}{x}'-\underset{\sim}{y}',\omega',\underset{\sim}{x}'',-\omega',\underset{\sim}{y}'')$$

$$\times S_0(\omega',\underset{\sim}{x}-\underset{\sim}{x}'-\underset{\sim}{x}'',\underset{\sim}{y}-\underset{\sim}{y}'-\underset{\sim}{y}'')d\underset{\sim}{x}'d\underset{\sim}{y}'d\underset{\sim}{x}''d\underset{\sim}{y}''d\omega' , \qquad (54)$$

where

$$H_0(\omega,\underset{\sim}{x}) = \int G_0(t,\underset{\sim}{x})e^{i\omega t}dt , \qquad (55)$$

$$H(\omega,\underset{\sim}{x}) = \int G(t,\underset{\sim}{x})e^{i\omega t}dt , \qquad (56)$$

$$\Theta(\omega,\underset{\sim}{\xi},\omega_+,\underset{\sim}{x},\omega_-,\underset{\sim}{y}) = \int\cdots\int \Phi(\tau,\underset{\sim}{\xi},t,\underset{\sim}{x},s,\underset{\sim}{y})$$

$$\times e^{i(\omega\tau+\omega_+ t+\omega_- s)}d\tau dt ds . \qquad (57)$$

The analysis leading to Eqs. 50, 51, 53, and 54 can be specialized further by considering again the case in which the operator L_1 can be written in the form given by Eq. 35, where now the μ_p's are stationary in space as well as in time, and the K_p's commute with both space and time translations. Then from Eq. 47 we have

$$F(t,\underset{\sim}{x},t-t',\ \underset{\sim}{x}-\underset{\sim}{x}') = \mu_p(t,\underset{\sim}{x})G_p(t-t',\underset{\sim}{x}-\underset{\sim}{x}')\ , \tag{58}$$

where

$$G_p(t,\underset{\sim}{x}) \equiv K_p G(t,\underset{\sim}{x})\ ;\quad p=1,\cdots,N\ . \tag{59}$$

Substituting the expression for F given by Eq. 58 into Eq. 52 yields

$$\Phi(\tau,\underset{\sim}{\xi},t,\underset{\sim}{x},s,\underset{\sim}{y}) = \Gamma_{pq}(\tau,\underset{\sim}{\xi})G_p(t,\underset{\sim}{x})G_q(s,\underset{\sim}{y})\ , \tag{60}$$

where

$$\Gamma_{pq}(\tau,\underset{\sim}{\xi}) \equiv \langle\mu_p(t,\underset{\sim}{x})\mu_q(t-\tau,\underset{\sim}{x}+\underset{\sim}{\xi})\rangle_A \tag{61}$$

for $p=1,\cdots,N$, $q=1,\cdots,N$. Upon combining Eqs. 57 and 60 we obtain

$$\Theta(\omega,\underset{\sim}{\xi},\omega_+,\underset{\sim}{x},\omega_-,\underset{\sim}{y}) = Z_{pq}(\omega,\underset{\sim}{\xi})$$

$$\times H_p(\omega_+,\underset{\sim}{x})H_q(\omega_-,\underset{\sim}{y})\ ;\ p=1,\cdots,N,\ q=1,\cdots,N\ , \tag{62}$$

where

$$Z_{pq}(\omega,\underset{\sim}{\xi}) = \int \Gamma_{pq}(\tau,\underset{\sim}{\xi})\,e^{i\omega\tau}d\tau\ ;\ p=1,\cdots,N,\ q=1,\cdots,N\ , \tag{63}$$

$$H_p(\omega,\underset{\sim}{x}) = \int G_p(t,\underset{\sim}{x})e^{i\omega t}dt;\ p=1,\cdots,N\ . \tag{64}$$

With the aid of Eqs. 51, 54, 60, and 62 we can write the formulas for $\tilde{R}$ and $\tilde{S}$ for this case in the form

$$\tilde{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \varepsilon^2 \int\cdots\int G_0(t',\underset{\sim}{x}')G_0(s',\underset{\sim}{y}')G_p(t'',\underset{\sim}{x}'')G_q(s'',\underset{\sim}{y}'')$$

$$\times \Gamma_{pq}(\tau - t' + s', \underset{\sim}{y} - \underset{\sim}{x} + \underset{\sim}{x}' - \underset{\sim}{y}')R_0(\tau - t' + s' - t'' + s'', \underset{\sim}{x} - \underset{\sim}{x}' - \underset{\sim}{x}'', \underset{\sim}{y} - \underset{\sim}{y}' - \underset{\sim}{y}'')$$

$$\times dt'd\underset{\sim}{x}'ds'd\underset{\sim}{y}'dt''d\underset{\sim}{x}''ds''d\underset{\sim}{y}'' , \tag{65}$$

$$\tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = (\varepsilon^2/2\pi)\int\cdots\int H_0(\omega,\underset{\sim}{x}')H_0^*(\omega,\underset{\sim}{y}')$$

$$\times \Big[Z_{pq}(\omega,\underset{\sim}{y} - \underset{\sim}{x} + \underset{\sim}{x}' - \underset{\sim}{y}') * H_p(\omega,\underset{\sim}{x}'')H_q^*(\omega,\underset{\sim}{y}'')$$

$$\times S_0(\omega,\underset{\sim}{x} - \underset{\sim}{x}' - \underset{\sim}{x}'', \underset{\sim}{y} - \underset{\sim}{y}' - \underset{\sim}{y}'')\Big] d\underset{\sim}{x}'d\underset{\sim}{y}'d\underset{\sim}{x}''d\underset{\sim}{y}'' . \tag{66}$$

The formulas for $\bar{R}$ and $\bar{S}$ for this case are given by Eqs. 50 and 53.

All of the formulas for R and S given in this section are accurate to order ε^2 ; i.e., the error in them is of order ε^3 . This is a consequence of the dropping of terms of order ε^3 in Eq. 3 and ε^2 in Eq. 4 (note that $\tilde{u}$ is of order ε) , and the vanishing of the cross terms in Eq. 17.

It should be pointed out that, for practical purposes, it is usually convenient to assume that all random processes under consideration are ergodic, as well as stationary, in time, in which case the average denoted by < > can be regarded as a time average.

II. APPLICATION

In order to show how the results obtained in the previous section may be applied to a particular case, we use them here to calculate the frequency spectrum of the scalar wave field radiated by a random point source in a three-dimensional time-dependent random medium. The starting point of this analysis is the scalar wave equation

$$(c^{-2}\partial_t^2 - \nabla^2)u = f \, , \tag{67}$$

where u is the wave function, f is the source term, and c is the local propagation speed of small disturbances of the medium. All quantities under consideration are assumed to be functions of t and $\underset{\sim}{x}$, where t is time and $\underset{\sim}{x}\left[=(x_1,x_2,x_3)\right]$ is a three-dimensional spatial coordinate.

Both the propagation speed c and the source term f are, in general, random functions; however, as discussed at the beginning of Sec. I, these functions are assumed to be statistically independent and are to be regarded as being dependent on parameters a and b, respectively, which are elements of different sample spaces A and B. In addition, we assume that the random fluctuations of the medium are stationary in time and space and represent a small perturbation of a uniform state. Thus we write

$$c(t,\underset{\sim}{x}) = c_0\left[1+\varepsilon\mu(t,\underset{\sim}{x})\right] \, , \tag{68}$$

where c_0 is a constant, ε is a small parameter which is a measure of the magnitude of the random fluctuations of the medium, and $\mu(t,\underset{\sim}{x})$ is a random function, stationary in t and $\underset{\sim}{x}$, with zero mean and unit variance; i.e., $\langle\mu\rangle_A = 0$, $\langle\mu^2\rangle_A = 1$.

As a consequence of the assumptions stated above, the relevant formulas for calculating the functions R and S for this case are given by Eqs. 50, 53, 65, and 66, along with Eqs. 22 and 26. (Actually, since we are interested here only in the function S, we need only be concerned with Eqs. 53, 66, and 26.) In order to make use of these equations we need to calculate the functions G_0, G, and G_1, as well as their transforms H_0, H, and H_1. We turn now to the calculation of these functions.

Upon substituting Eq. 68 into Eq. 67 and expanding in powers of ε we can write the latter equation in the form

$$(L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \cdots)u = f , \tag{69}$$

where the operators L_0, L_1, and L_2 are given by

$$L_0 = c_0^{-2}\partial_t^2 - \nabla^2 , \tag{70}$$

$$L_1 = -2c_0^{-2}\mu\partial_t^2 , \tag{71}$$

$$L_2 = 3c_0^{-2}\mu^2\partial_t^2 . \tag{72}$$

The function G_0 for this case, which corresponds to a spherical pulsed wave propagating in a uniform medium, is obtained by solving Eq. 45, with L_0 given by Eq. 70, subject to the initial condition $G_0 = 0$ for $t < 0$. This yields the familiar waveform given by

$$G_0(t,\underset{\sim}{x}) = (4\pi x)^{-1}\delta(t - c_0^{-1}x) . \tag{73}$$

By inserting the expression for $G_0(t - t', \underset{\sim}{x} - \underset{\sim}{x}')$, as given by Eq. 73, into Eq. 7, carrying out the integration over t', and changing the spatial integration variable, we can express the operator L_0^{-1} in the form

$$L_0^{-1}w(t,\underset{\sim}{x}) = (4\pi)^{-1}\int \xi^{-1}w(t-c_0^{-1}\xi,\underset{\sim}{x}+\underset{\sim}{\xi})d\underset{\sim}{\xi}\,. \tag{74}$$

The function $H_0(\omega,\underset{\sim}{x})$ for this case is easily obtained by transforming Eq. 73 according to Eq. 55. The result is

$$H_0(\omega,\underset{\sim}{x}) = (4\pi x)^{-1}e^{ikx}\,, \tag{75}$$

where $k=\omega/c_0$.

The function $G(t,\underset{\sim}{x})$ for this case is determined by Eq. 46, where the operator M is given by Eq. 5. With the aid of these equations, along with Eqs. 70, 71, 72, and 74, we can write the equation for $G(t,\underset{\sim}{x})$ in the form

$$\begin{aligned}(c_0^{-2}\partial_t^2-\nabla^2)G(t,\underset{\sim}{x}) &+ \varepsilon^2\Big\{3c_0^{-2}G_{tt}(t,\underset{\sim}{x}) - (\pi c_0^4)^{-1} \\ &\times\int\xi^{-1}\Big[\Gamma(c_0^{-1}\xi,\underset{\sim}{\xi})G_{tttt}(t-c_0^{-1}\xi,\underset{\sim}{x}+\underset{\sim}{\xi}) - 2\Gamma_\tau(c_0^{-1}\xi,\underset{\sim}{\xi}) \\ &\times G_{ttt}(t-c_0^{-1}\xi,\underset{\sim}{x}+\underset{\sim}{\xi}) + \Gamma_{\tau\tau}(c_0^{-1}\xi,\underset{\sim}{\xi})G_{tt}(t-c_0^{-1}\xi,\underset{\sim}{x}+\underset{\sim}{\xi})\Big]d\underset{\sim}{\xi}\Big\} \\ &= \delta(t)\delta(\underset{\sim}{x})\,,\end{aligned} \tag{76}$$

where the subscripts denote derivatives, and we have defined

$$\Gamma(\tau,\underset{\sim}{\xi}) = \langle\mu(t,\underset{\sim}{x})\mu(t-\tau,\underset{\sim}{x}+\underset{\sim}{\xi})\rangle_A\,. \tag{77}$$

The initial condition for G is that $G=0$ for $t<0$.

The procedure by which Eq. 76 is solved for $G(t,\underset{\sim}{x})$ is described in Appendix A. Since we wish only to calculate the function S, we need only the transform $H(\omega,\underset{\sim}{x})$ of $G(t,\underset{\sim}{x})$, as defined by Eq. 56. For the case in which the medium is statistically isotropic [i.e., when $\Gamma(\tau,\underset{\sim}{\xi})=\Gamma(\tau,\xi)$], this quantity is given by Eq. A11, which we rewrite here in the form

$$H(\omega,\underset{\sim}{x}) = C(k)(4\pi x)^{-1} e^{i\kappa x} , \tag{78}$$

where

$$C(k) = 1 + 2\varepsilon^2 \psi(k) . \tag{79}$$

The function $\psi(k)$ is given by Eq. A12, and κ is given by Eq. A10.

The functions $G_1(t,\underset{\sim}{x})$ and $H_1(t,\underset{\sim}{x})$ are determined by Eqs. 59 and 64, where, for the case under consideration, $N = 1$ and, in view of Eqs. 35 and 71, $\mu_1(t,\underset{\sim}{x}) = \mu(t,\underset{\sim}{x})$ and $K_1 = -2c_0^{-2}\partial_t^2$. It follows that

$$G_1(t,\underset{\sim}{x}) = -2c_0^{-2} G_{tt}(t,\underset{\sim}{x}) , \tag{80}$$

and hence that

$$H_1(\omega,\underset{\sim}{x}) = 2k^2 H(\omega,\underset{\sim}{x}) . \tag{81}$$

The source term f is assumed to represent a point source in space but one which is random in time. Accordingly we write

$$f(t,\underset{\sim}{x}) = g(t)\delta(\underset{\sim}{x}) , \tag{82}$$

where $g(t)$ is a stationary random function with zero mean. Equation 21 then yields

$$R_0(\tau,\underset{\sim}{x},\underset{\sim}{y}) = P_0(\tau)\delta(\underset{\sim}{x})\delta(\underset{\sim}{y}) , \tag{83}$$

where $P_0(\tau)$ is defined by

$$P_0(\tau) = \langle g(t)g(t-\tau)\rangle_B . \tag{84}$$

Upon transforming Eq. 83 according to Eq. 31 we obtain

$$S_0(\omega,\underset{\sim}{x},\underset{\sim}{y}) = Q_0(\omega)\delta(\underset{\sim}{x})\delta(\underset{\sim}{y}) , \tag{85}$$

where we have defined

$$Q_0(\omega) = \int P_0(\tau) e^{i\omega\tau} d\tau . \tag{86}$$

Expressions for $\bar{S}$ and $\tilde{S}$ can now be obtained by substituting the formula for S_0 given by Eq. 85 into Eqs. 53 and 66 and carrying out the integration over $\mathbf{x}'$ and $\mathbf{y}'$ in Eq. 53 and over $\mathbf{x}''$ and $\mathbf{y}''$ in Eq. 66. The result is

$$\bar{S}(\omega,\mathbf{x},\mathbf{y}) = Q_0(\omega)H(\omega,\mathbf{x})H^*(\omega,\mathbf{y}) \ , \tag{87}$$

$$\tilde{S}(\omega,\mathbf{x},\mathbf{y}) = (2\varepsilon^2/\pi c_0^4)\iint H_0(\omega,\mathbf{x}')H_0^*(\omega,\mathbf{y}')$$

$$\times \left[Z(\omega,\mathbf{y}-\mathbf{x}+\mathbf{x}'-\mathbf{y}') * \omega^4 Q_0(\omega)H(\omega,\mathbf{x}-\mathbf{x}')H^*(\omega,\mathbf{y}-\mathbf{y}')\right] d\mathbf{x}'d\mathbf{y}' \ , \tag{88}$$

where Z is given by

$$Z(\omega,\boldsymbol{\xi}) = \int \Gamma(\tau,\boldsymbol{\xi})e^{i\omega\tau}d\tau \ , \tag{89}$$

and we have used Eq. 81 to substitute for H_1 in terms of H. The spectral density function $S(\omega,\mathbf{x},\mathbf{y})$ can now be calculated in terms of known functions with the aid of Eqs. 26 , 87, 88, 75, and 78.

The expression for $\bar{S}$ and $\tilde{S}$ given by Eqs. 87 and 88 can be considerably simplified in the case of high-frequency waves; i.e., when the characteristic frequency of the source is much greater than that of the medium. In considering this case we shall restrict our attention to the power spectrum $Q(\omega,\mathbf{x})$, which is defined by

$$Q(\omega,\mathbf{x}) = S(\omega,\mathbf{x},\mathbf{x}) \ . \tag{90}$$

From Eq. 26 we have

$$Q(\omega,\mathbf{x}) = \bar{Q}(\omega,\mathbf{x}) + \tilde{Q}(\omega,\mathbf{x}) \ , \tag{91}$$

where

$$\bar{Q}(\omega,\mathbf{x}) \equiv \bar{S}(\omega,\mathbf{x},\mathbf{x}) \ , \tag{92}$$

$$\tilde{Q}(\omega,\underset{\sim}{x}) \equiv \tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{x}) \; . \tag{93}$$

Eqs. 87 and 88 yield

$$\bar{Q}(\omega,\underset{\sim}{x}) = Q_0(\omega)\;|H(\omega,\underset{\sim}{x})|^2 \; , \tag{94}$$

$$\tilde{Q}(\omega,\underset{\sim}{x}) = (2\varepsilon^2/\pi c_0^4)\iint H_0(\omega,\underset{\sim}{x}')H_0^*(\omega,\underset{\sim}{y}')$$

$$\times\left[Z(\omega,\underset{\sim}{x}'-\underset{\sim}{y}') * \omega^4 Q_0(\omega)H(\omega,\underset{\sim}{x}-\underset{\sim}{x}')H^*(\omega,\underset{\sim}{x}-\underset{\sim}{y}')\right]d\underset{\sim}{x}'d\underset{\sim}{y}' \; . \tag{95}$$

After changing the integration variables in Eq. 95 we can write

$$\tilde{Q}(\omega,\underset{\sim}{x}) = (2\varepsilon^2/\pi c_0^4)\iint H_0(\omega,\underset{\sim}{x}-\underset{\sim}{x}')H_0^*(\omega,\underset{\sim}{x}-\underset{\sim}{x}'')$$

$$\times\left[Z(\omega,\underset{\sim}{x}''-\underset{\sim}{x}') * \omega^4 Q_0(\omega)H(\omega,\underset{\sim}{x}')H^*(\omega,\underset{\sim}{x}'')\right]d\underset{\sim}{x}'d\underset{\sim}{x}'' \; . \tag{96}$$

The first step in the high-frequency analysis is to obtain an asymptotic expansion, valid for large k, for the quantity κ. This is easily accomplished by integrating by parts in Eq. A10, after substituting for χ from Eq. A2 and recalling that $\Gamma(\tau,\underset{\sim}{\xi}) = \Gamma(\tau,\xi)$ for the isotropic case under consideration. This yields the approximation

$$\kappa \simeq k + i\alpha \; , \tag{97}$$

where

$$\alpha = \varepsilon^2 k^2 \ell \; . \tag{98}$$

The quantity ℓ is a characteristic length scale associated with the medium, and is defined by

$$\ell = \int_0^\infty \Gamma(c_0^{-1}\xi,\xi)d\xi \; . \tag{99}$$

With the aid of Eqs. 75, 78, and 97 we can write Eqs. 94 and 96 in the form

$$\bar{Q}(\omega,\underset{\sim}{x}) = Q_0(\omega)\,|C(k)^2|\,(4\pi x)^{-2} e^{-2\alpha x} \;, \tag{100}$$

$$\begin{aligned}\tilde{Q}(\omega,\underset{\sim}{x}) = \frac{2\varepsilon^2}{\pi(4\pi)^4}\iint & \frac{e^{ik(|\underset{\sim}{x}-\underset{\sim}{x}'| - |\underset{\sim}{x}-\underset{\sim}{x}''|)}}{|\underset{\sim}{x}-\underset{\sim}{x}'|\,x'\,|\underset{\sim}{x}-\underset{\sim}{x}''|\,x''} \\ & \times \Big[Z(\omega,\underset{\sim}{x}''-\underset{\sim}{x}') * k^4 |C(k)|^2 Q_0(\omega) e^{ik(x'-x'')} \\ & \times e^{-\alpha(x'+x'')} \Big] d\underset{\sim}{x}'\,d\underset{\sim}{x}'' \;,\end{aligned} \tag{101}$$

where, from Eq. 79 ,

$$|C(k)|^2 = 1 + 4\varepsilon^2 \mathrm{Re}\psi(k) \;. \tag{102}$$

(In deriving Eq. 102 terms of order ε^4 were dropped.)

The integral over $\underset{\sim}{x}'$ and $\underset{\sim}{x}''$ in Eq. 101 has been evaluated using the forward-scatter approximation. The details of that calculation are given in Appendix B. The resulting approximate expression for $\tilde{Q}(\omega,\underset{\sim}{x})$ can be written

$$\tilde{Q}(\omega,\underset{\sim}{x}) = (4\pi x)^{-2}\Big[W(\omega) * |C(k)|^2 (1 - e^{-2\alpha x}) Q_0(\omega) \Big] \;, \tag{103}$$

where W is defined by

$$W(\omega) = (4\pi\ell)^{-1} \hat{Z}(\omega,k) \;, \tag{104}$$

and $\hat{Z}$ is given by Eq. B17.

An expression for $Q(\omega,\underset{\sim}{x})$ can now be obtained by substituting the formulas for $\bar{Q}(\omega,\underset{\sim}{x})$ and $\tilde{Q}(\omega,\underset{\sim}{x})$ given by Eqs. 100 and 103 into Eq. 91.

In so doing we simplify matters slightly by making the approximation $|C(k)|^2 = 1$. After dividing through by the spherical-spreading term $(4\pi x)^{-2}$ we obtain finally

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) = e^{-2\alpha x} Q_0(\omega) + \left[W(\omega) * (1 - e^{-2\alpha x}) Q_0(\omega)\right] . \tag{105}$$

It should be pointed out here that, although the error in the general formulas for R and S given in Sec. I is of order ε^3, the error in Eq. 105 is of order ε^2. This is because some terms of order ε^2 were dropped in the derivation of this equation.

We see from Eq. 105 that, as $\alpha x \to 0$,

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) \to Q_0(\omega) . \tag{106}$$

Thus, as ε and/or x (the source-field point distance) goes to zero, the wave spectrum (with the spherical-spreading term factored out) approaches the source spectrum, as we would expect. In the opposite limit, i.e., as $\alpha x \to \infty$, Eq. 105 shows that

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) \to W * Q_0(\omega) . \tag{107}$$

We see therefore that the wave spectrum (again apart from the spherical-spreading term) tends to a limiting form as $x \to \infty$. This limiting form, which is given by the right-hand side of Eq. 107, is referred to here as the fully-developed spectrum.

It may be verified by direct integration of Eq. 105 that

$$(4\pi x)^2 \int Q(\omega,\underset{\sim}{x}) d\omega = \int Q_0(\omega) d\omega . \tag{108}$$

In the derivation of Eq. 108 we have used the fact that

$$\int W(\omega)\, d\omega = 1\ . \tag{109}$$

Equation 108 shows that the total signal power; i.e., the area under the spectral curve, normalized by the spherical-spreading term, is conserved.

We can simplify Eq. 105 further by assuming that the band-width of the source is so small that, insofar as the convolution integral is concerned, $Q_0(\omega)$ can be regarded as a delta function. Accordingly we replace $Q_0(\omega)$ in the convolution term by

$$A_0\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

(since Q_0 must be an even function), where $A_0>0$ and $\omega_0>0$. We can also write, in this case,

$$e^{-2\alpha x}Q_0(\omega) \simeq e^{-2\alpha_0 x}Q_0(\omega)\ ,$$

where $\alpha_0=\varepsilon^2 k_0^2 \ell$ and $k_0=\omega_0/c_0$. Then Eq. 105 becomes

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) = e^{-2\alpha_0 x}Q_0(\omega) + (1-e^{-2\alpha_0 x})Q_\infty(\omega)\ , \tag{110}$$

where

$$Q_\infty(\omega) = A_0\left[W(\omega-\omega_0)+W(\omega+\omega_0)\right]\ . \tag{111}$$

Note that, as a consequence of Eq. 108, we must have

$$\int Q_\infty(\omega)\, d\omega = \int Q_0(\omega)\, d\omega\ , \tag{112}$$

and hence, in view of Eq. 109,

$$A_0 = \tfrac{1}{2}\int Q_0(\omega)d\omega \quad . \tag{113}$$

By introducing a "broadening parameter" β, defined by

$$\beta = 1 - e^{-2\alpha_0 x} , \tag{114}$$

we can write Eq. 110 in the form

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) = (1-\beta)Q_0(\omega) + \beta Q_\infty(\omega) \ . \tag{115}$$

Thus we see that the wave spectrum (with the spherical-spreading term factored out) can be regarded in this case as a linear (in β) interpolation between the source spectrum $Q_0(\omega)$ and the fully-developed spectrum $Q_\infty(\omega)$.

In order to show the broadening phenomenon graphically, numerical calculations of the quantity $(4\pi x)^2 Q(\omega,\underset{\sim}{x})$ as a function of ω have been made for various values of β using Eq. 115. For this purpose the source spectrum $Q_0(\omega)$ was chosen to be a Gaussian function, sharply peaked about a frequency ω_0 called the carrier frequency. The function $W(\omega)$ was also chosen to be a Gaussian, less sharply peaked, centered in the first instance about $\omega = 0$ and in the second about a frequency Ω_0 for which $0 < \Omega_0 << \omega_0$.

The results of these calculations are plotted (in dimensionless coordinates) in Figs. 1 and 2. All of the curves in each figure are plotted on the same scale. Note that in each figure the curve labeled $\beta = 0$ corresponds to the source spectrum, the curve labeled $\beta = 1$ corresponds to the fully-developed spectrum, and those labeled with values of β between zero and one correspond to intermediate stages in the broadening process. Both sets of curves show clearly the broadening of the wave spectrum with increasing β. The two sets differ, however, in one respect. The results shown in Fig. 2, for which the function $W(\omega)$ is peaked at a non-zero value of ω, are marked by the appearance of side lobes on the broadened spectrum. In Fig. 1, by contrast, for which $W(\omega)$ is peaked at $\omega = 0$, no such side lobes appear.

The results obtained here appear generally to be in qualitative agreement with observation, as can be seen by, for example, comparing Fig. 1 with Fig. 11 of Ref. 1 or Fig. 3 of Ref. 2. Note moreover that the observations reported in Refs. 1 and 2 indicate conservation of total signal power, which is also consistent with the present results (cf. Eq. 108).

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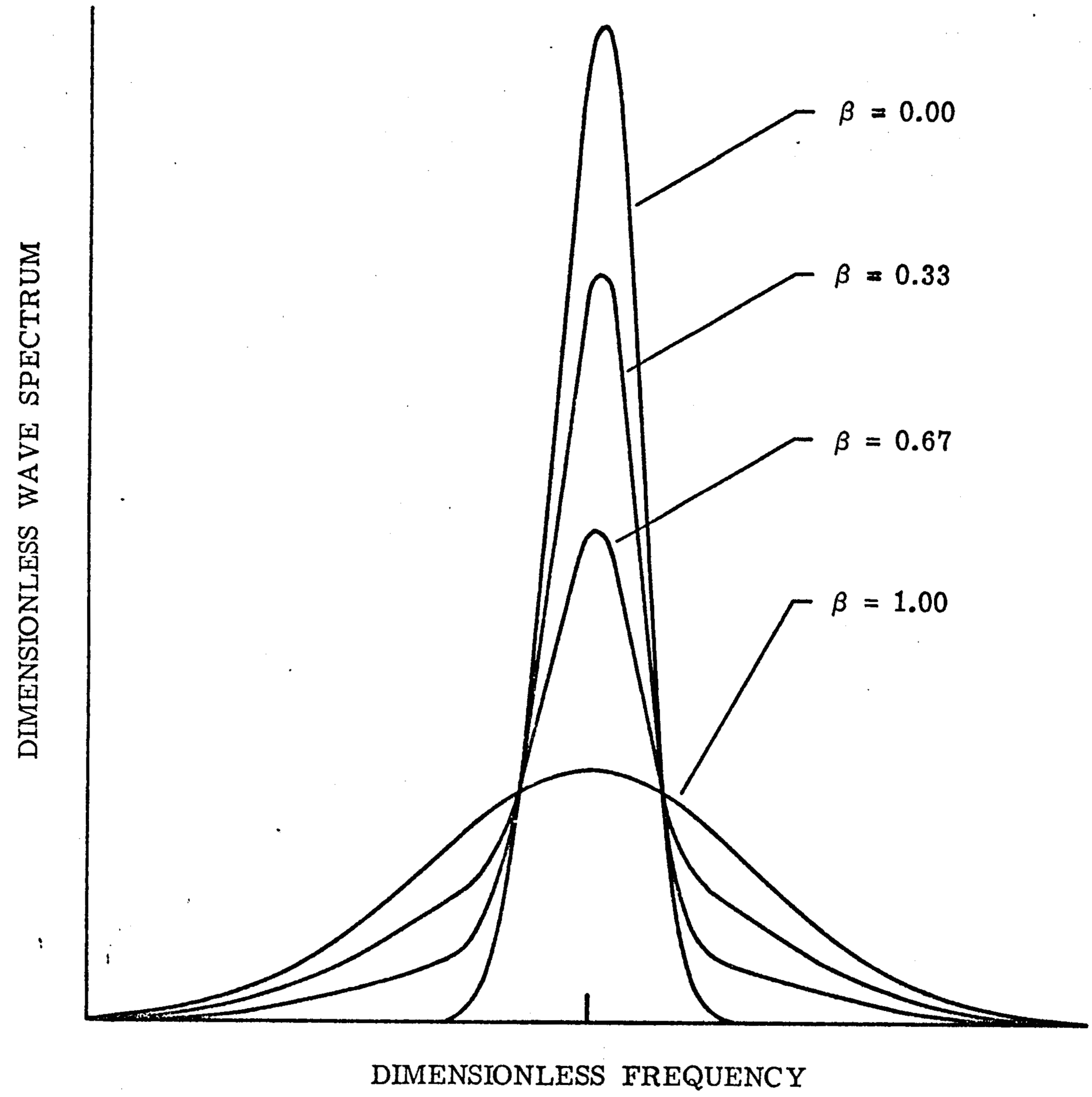

β = 0.00
β = 0.33
β = 0.67
β = 1.00
DIMENSIONLESS WAVE SPECTRUM
DIMENSIONLESS FREQUENCY

Fig. 1

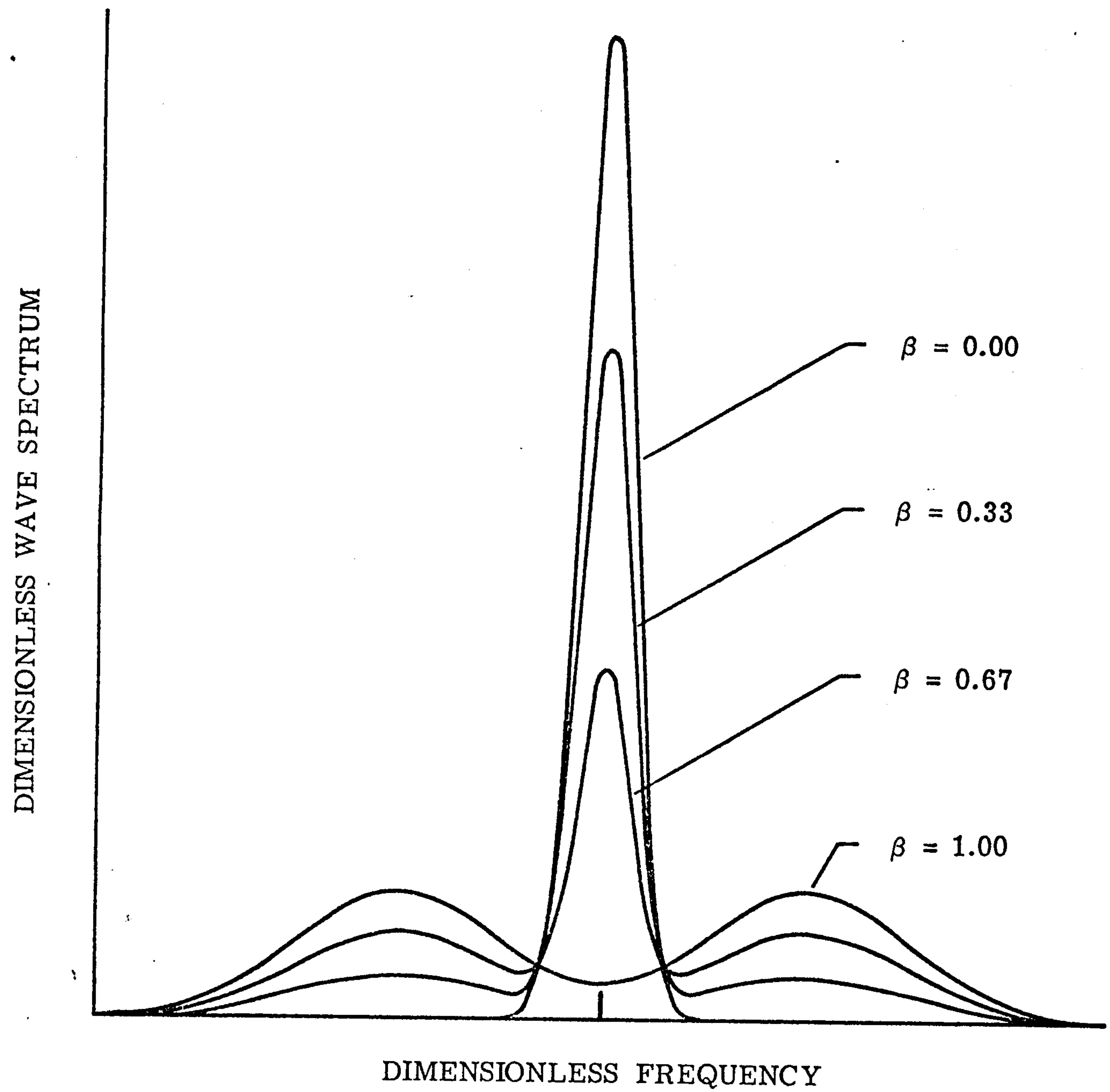
DIMENSIONLESS WAVE SPECTRUM
β = 0.00
β = 0.33
β = 0.67
β = 1.00
DIMENSIONLESS FREQUENCY

Fig. 2

APPENDIX A. CALCULATION OF $G(t,\underset{\sim}{x})$ AND $H(\omega,\underset{\sim}{x})$

The function $G(t,\underset{\sim}{x})$ of Sec. 2 is determined by Eq. 76, together with the initial condition $G = 0$ for $t < 0$. An equation for the transform $H(\omega,\underset{\sim}{x})$ of $G(t,\underset{\sim}{x})$, as defined by Eq. 56, is obtained by transforming both sides of Eq. 76. The result is

$$\left[\nabla^2 + (1+3\epsilon^2)k^2\right]H(\omega,\underset{\sim}{x}) + (\epsilon^2k^2/\pi)\int \xi^{-1}e^{ik\xi}$$

$$\times\, \chi(k,\underset{\sim}{\xi})H(\omega,\underset{\sim}{x}+\underset{\sim}{\xi})d\underset{\sim}{\xi} = -\delta(\underset{\sim}{x}) \,, \qquad \text{(A1)}$$

where the function $\chi(k,\underset{\sim}{\xi})$ is defined by

$$\chi(k,\underset{\sim}{\xi}) = k^2\Gamma(c_0^{-1}\xi,\underset{\sim}{\xi}) - 2ikc_0^{-1}\Gamma_\tau(c_0^{-1}\xi,\underset{\sim}{\xi})$$

$$- c_0^{-2}\Gamma_{\tau\tau}(c_0^{-1}\xi,\underset{\sim}{\xi}) \,. \qquad \text{(A2)}$$

In order to solve Eq. A1 we introduce the spatial Fourier transform $\hat{H}(\omega,\underset{\sim}{m})$ of $H(\omega,\underset{\sim}{x})$, defined by

$$\hat{H}(\omega,\underset{\sim}{m}) = \int H(\omega,\underset{\sim}{x})e^{-i\underset{\sim}{m}\cdot\underset{\sim}{x}}d\underset{\sim}{x} \,, \qquad \text{(A3)}$$

where $\underset{\sim}{m}\cdot\underset{\sim}{x} \equiv \sum_{i=1}^{3} m_i x_i$. Transforming both sides of Eq. A1 according to the prescription given by Eq. A3 and solving for $\hat{H}$ yields

$$\hat{H}(\omega,\underset{\sim}{m}) = [D(k,\underset{\sim}{m})]^{-1} \,, \qquad \text{(A4)}$$

where we have defined

$$D(k,\underset{\sim}{m}) = m^2 - (1+3\epsilon^2)k^2 - (\epsilon^2k^2/\pi)\int \xi^{-1}e^{ik\xi}$$

$$\times\, \chi(k,\underset{\sim}{\xi})e^{i\underset{\sim}{m}\cdot\underset{\sim}{\xi}}d\underset{\sim}{\xi} \,. \qquad \text{(A5)}$$

With the aid of Eq. A4 we can now express $H(\omega,\underset{\sim}{x})$ as an inverse transform; i.e., we write

$$H(\omega,\underset{\sim}{x}) = (8\pi^3)^{-1} \int \left[D(k,\underset{\sim}{m})\right]^{-1} e^{i\,\underset{\sim}{m}\cdot\underset{\sim}{x}} d\underset{\sim}{m} \quad . \tag{A6}$$

In order to proceed further we assume that the medium is statistically isotropic, so that we can write (in an obvious change of notation) $\Gamma(\tau,\underset{\sim}{\xi}) = \Gamma(\tau,\xi)$. Then, in view of Eq. A2, we can also write $\chi(k,\underset{\sim}{\xi}) = \chi(k,\xi)$. The angular integration in Eq. A5 can now be carried out, yielding

$$D(k,\underset{\sim}{m}) = D(k,m) = m^2 - (1+3\varepsilon^2)k^2 - 4\varepsilon^2 k^2 m^{-1} \int_0^\infty e^{ik\xi} \chi(k,\xi) \sin m\xi d\xi \, . \tag{A7}$$

Upon substituting the expression for D given by Eq. A7 into Eq. A6 and carrying out the angular integration we find that

$$H(\omega,\underset{\sim}{x}) = (2\pi^2 x)^{-1} \int_0^\infty [D(k,m)]^{-1} m \sin mx \, dm \, . \tag{A8}$$

The integral in Eq. A8 can be evaluated by means of contour integration, after which the expression for H can be written

$$H(\omega,\underset{\sim}{x}) = (2\pi x)^{-1} [D_m(k,\kappa)]^{-1} \kappa e^{i\kappa x} \, . \tag{A9}$$

Here D_m denotes the derivative of $D(k,m)$ with respect to m (regarded now as a complex variable), and κ is the root of the dispersion equation $D(k,\kappa) = 0$ which has the property that $\kappa \to k$ as $\varepsilon \to 0$. To lowest order in ε, this root is given by

$$\kappa = k\left\{1 + \tfrac{1}{2}\varepsilon^2\left[3 + 4k^{-1}\int_0^\infty e^{ik\xi}\chi(k,\xi)\sin k\xi \, d\xi\right]\right\}. \tag{A10}$$

Upon substituting the expression for κ given by Eq. A10 into Eq. A9, after calculating D_m using Eq. A7, we obtain

$$H(\omega,\underset{\sim}{x}) = \left[1 + 2\varepsilon^2\psi(k)\right](4\pi x)^{-1}e^{i\kappa x}, \tag{A11}$$

where

$$\psi(k) = \int_0^\infty e^{ik\xi}\chi(k,\xi)\left(\cos k\xi - \frac{\sin k\xi}{k\xi}\right)\xi d\xi, \tag{A12}$$

and higher-order terms in ε have been dropped.

The function $G(t,\underset{\sim}{x})$ can now be obtained by applying the inverse Fourier transform to Eq. A11. We shall not carry out this calculation here, however, since we need only the function $H(\omega,\underset{\sim}{x})$. This calculation was carried out in Ref. 21 for the case of a time-independent medium.

APPENDIX B. CALCULATION OF $\tilde{Q}$ USING THE FORWARD-SCATTER APPROXIMATION

By making explicit the ω' integration in Eq. 101 and changing the order of integration we can write the expression for $\tilde{Q}$ in the form

$$\tilde{Q}(\omega,\underset{\sim}{x}) = \left[2\varepsilon^2/\pi(4\pi)^4\right]\int k'^4|C(k')|^2 Q_0(\omega')I(\omega,\underset{\sim}{x};\omega')d\omega' , \tag{B1}$$

where the integral I is defined by

$$I(\omega,\underset{\sim}{x};\omega') = \iint \frac{e^{ik(|\underset{\sim}{x}-\underset{\sim}{x}'| - |\underset{\sim}{x}-\underset{\sim}{x}''|)}}{|\underset{\sim}{x}-\underset{\sim}{x}'|x'|\underset{\sim}{x}-\underset{\sim}{x}''|x''} Z(\omega-\omega',\underset{\sim}{x}''-\underset{\sim}{x}')$$

$$\times e^{ik'(x'-x'')}e^{-\alpha'(x'+x'')}d\underset{\sim}{x}'d\underset{\sim}{x}'' . \tag{B2}$$

Here $k' = \omega'/c_0$ and $\alpha' = \varepsilon^2 k'^2 \ell$. Equation B2 can be written in the alternate form

$$I(\omega,\underset{\sim}{x};\omega') = \iint \frac{e^{ik'(|\underset{\sim}{x}-\underset{\sim}{x}'|+x')}}{|\underset{\sim}{x}-\underset{\sim}{x}'|x'} \frac{e^{-ik'(|\underset{\sim}{x}-\underset{\sim}{x}''|+x'')}}{|\underset{\sim}{x}-\underset{\sim}{x}''|x''}$$

$$\times Z(\omega-\omega',\underset{\sim}{x}''-\underset{\sim}{x}')e^{i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}'|}e^{-i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}''|}$$

$$\times e^{-\alpha'(x'+x'')}d\underset{\sim}{x}'d\underset{\sim}{x}'' , \tag{B3}$$

which is more convenient for the application of the forward-scatter approximation.

We begin the analysis by substituting for Z in terms of Γ in Eq. B3 with the aid of Eq. 89 . By changing the order of integration in the resulting expression for I we get

$$I = \int e^{i(\omega-\omega')\tau} \iint \frac{e^{ik'(|\underset{\sim}{x}-\underset{\sim}{x}'|+x')}}{|\underset{\sim}{x}-\underset{\sim}{x}'|x'} \frac{e^{-ik'(|\underset{\sim}{x}-\underset{\sim}{x}''|+x'')}}{|\underset{\sim}{x}-\underset{\sim}{x}''|x''}$$

$$\times \Gamma(\tau,\underset{\sim}{x}''-\underset{\sim}{x}')e^{i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}'|}e^{-i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}''|}$$

$$\times e^{-\alpha'(x'+x'')}d\underset{\sim}{x}'d\underset{\sim}{x}''d\tau \ . \tag{B4}$$

Next we use Eq. 77 to substitute for Γ in terms of μ in Eq. B4. Upon reversing the order of the averaging and integration (over $\underset{\sim}{x}'$ and $\underset{\sim}{x}''$) processes, we note that the double spatial integral can be split into a product of two integrals. Equation B4 can then be written

$$I = \int e^{i(\omega-\omega')\tau} \langle J\rangle_A d\tau \ , \tag{B5}$$

where

$$J = J_+J_- \ , \tag{B6}$$

$$J_+ = \int \frac{e^{ik'(|\underset{\sim}{x}-\underset{\sim}{x}'|+x')}}{|\underset{\sim}{x}-\underset{\sim}{x}'|x'} \mu(t,\underset{\sim}{x}')e^{i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}'|}e^{-\alpha'x'}d\underset{\sim}{x}', \tag{B7}$$

and

$$J_- = \int \frac{e^{-ik'(|\underset{\sim}{x}-\underset{\sim}{x}''|+x'')}}{|\underset{\sim}{x}-\underset{\sim}{x}''|x''} \mu(t-\tau,\underset{\sim}{x}'')e^{-i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}''|}$$

$$\times e^{-\alpha'x''}d\underset{\sim}{x}'' \ . \tag{B8}$$

We can now apply the forward-scatter approximation, as discussed in Ref. 22, to the integrals J_+ and J_-. This yields

$$J_+ = (2\pi i/k'x)e^{ik'x}\int_0^x \mu(t,0,0,x')e^{i(k-k')(x-x')}$$

$$\times e^{-\alpha' x'}dx' + O(k'^{-2}) \ , \tag{B9}$$

$$J_- = -(2\pi i/k'x)e^{-ik'x}\int_0^x \mu(t-\tau,0,0,x'')e^{-i(k-k')(x-x'')}$$

$$\times e^{-\alpha' x''}dx'' + O(k'^{-2}) \ . \tag{B10}$$

In the derivation of Eqs. B9 and B10 we have set $\underset{\sim}{x} = (0,0,x)$. This entails no loss of generality since the medium has been assumed statistically isotropic.

Conditions for the validity of the forward-scatter approximation are given in Ref. 22. In the present context these conditions take the form

$$k_1^{-1} \ll x \ll k_1 \ell^2 \ , \tag{B11}$$

where k_1 is a characteristic wavenumber associated with the wave field.

By substituting the expressions for J_+ and J_- given by Eqs. B9 and B10 into Eq. B6, dropping terms of order k'^{-3}, and averaging, we obtain

$$\langle J \rangle_A = (4\pi^2/k'^2x^2)\int_0^x\int_0^x \Gamma(\tau,x''-x')e^{i(k-k')(x''-x')}$$

$$\times e^{-\alpha'(x''+x')}dx'dx'' \ . \tag{B12}$$

The double integral in Eq. B12 can be partially evaluated with the aid of the coordinate transformation $\xi = x'' - x'$, $\eta = x'' + x'$. The result is

$$\langle J\rangle_A = (4\pi^2/\alpha' k'^2 x^2)\int_0^x \Gamma(\tau,\xi)\,(e^{-\alpha'\xi} - e^{-2\alpha' x}e^{\alpha'\xi})$$

$$\times \cos[(k-k')\xi]\,d\xi \;. \tag{B13}$$

In deriving Eq. B13 we have made use of the fact that $\Gamma(\tau,\xi)$ is even in ξ.

We can now get a series expansion for $\langle J\rangle_A$ in powers of α' (which is equivalent to an expansion in powers of ε^2) by expanding the terms $\exp(\alpha'\xi)$ and $\exp(-\alpha'\xi)$ in Eq. B13 and integrating term by term. This yields

$$\langle J\rangle_A = (4\pi^2/\alpha' k'^2 x^2)\sum_{n=0}^{\infty}(-1)^n\left[1-(-1)^n e^{-2\alpha' x}\right](\alpha'^n/n!)$$

$$\times \int_0^x \xi^n\,\Gamma(\tau,\xi)\cos[(k-k')\xi]\,d\xi \;. \tag{B14}$$

When $x \gg \ell$ the integration in Eq. B14 can be extended to $+\infty$ without introducing significant error into the integral. Upon dropping all but the first term of the resulting expansion we obtain

$$\langle J\rangle_A \simeq (4\pi^2/\alpha' k'^2 x^2)(1-e^{-2\alpha' x})\int_0^{\infty}\Gamma(\tau,\xi)\cos[(k-k')\xi]d\xi \;. \tag{B15}$$

An approximate expression for the integral I can now be obtained by substituting the result for $\langle J\rangle_A$ given by Eq. B15 into Eq. B5 and carrying out the integration over τ. This yields

$$I = (2\pi^2/\alpha' k'^2 x^2)(1-e^{-2\alpha' x})\,\hat{Z}(\omega-\omega', k-k') \;, \tag{B16}$$

where we have defined

$$\hat{Z}(\omega,\nu) = 2\int_0^{\infty} Z(\omega,\xi)\cos\nu\xi\,d\xi \;. \tag{B17}$$

Upon combining Eqs. B1 and B16 we obtain the expression for $\tilde{Q}$ given by Eq. 103.